Low-Frequency Behavior of the Propagation Constant Along a Thin Wire in an Arbitrarily Shaped Mine Tunnel

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***Abstract*—Seidel and Wait have investigated the complex propagation constant (phase and attenuation coefficients) of the fundamental mode of propagation for radio waves along a thin wire or cable, located in an elliptical mine tunnel, and found that the attenuation rate for low frequency is insensitive to the shape of the ellipse if the cable-wall distance and cross-sectional area are kept constant. We consider here tunnels of more general cross section, and obtain a characteristic equation for the propagation constant, valid for sufficiently low frequency, by means of a variational formulation of an integral equation. The characteristic equation involves only the electrical parameters of the tunnel walls, the radius of the wire, and the capacitance per unit length that the wire would have if the tunnel walls were perfectly conducting. Agreement with exact calculations for several geometries is found to be excellent below about 100 kHz, and acceptable even up to 1 MHz or more, for typical tunnel parameters. Since the wire capacitance can be shown to depend most importantly on its distance from the wall and on the area of the tunnel, the conclusion of Seidel and Wait can be made more precise and extended to tunnels of arbitrary cross section.**

I. Introduction

MUCH INTEREST has been shown recently in the attenuation and propagation constants of waves propagating along wires, cables, rails, or other such structures in mine tunnels, because of the extensive use to which these have been put for communications purposes. When the adjacent rock which forms the tunnel wall has finite conductivity, it is necessary to idealize the problem by assuming some simple cross-sectional shape (usually circular) for the tunnel in order to formulate the problem exactly [1]–[3]. Recently, Seidel and Wait [4] investigated the case of an elliptical mine tunnel in order to determine the effect of a noncircular cross section on the attenuation constant. It was found that the attenuation rate at low frequency is insensitive to the eccentricity of the ellipse if the cable-wall distance and cross-sectional area are kept constant. It might reasonably be speculated that this result does not depend on the specific shape of the tunnel involved, but only on some more general property of the tunnel.

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In this work, tunnels of arbitrary shape will be considered, and a characteristic equation for the propagation constant and attenuation rate at low frequencies will be obtained by means of a variational formulation of an integral equation for the electric field of the transmission-line mode. This equation involves only the electrical parameters of the tunnel walls, the radius of the wire, and the capacitance per unit length the wire would have if the tunnel walls were perfectly conducting. Comparisons of numerical results with those of more exact computations are presented.

II. Low-Frequency Characteristic Equation

A method for determining approximately the propagation constant at low frequency of a mode along a thin wire located within a mine tunnel of arbitrary cross section can be found by starting with Katsenelenbaum's [5] integral equation for the fields of a mode on an arbitrarily shaped dielectric rod. The tunnel geometry is indicated in Fig. 1. The medium external to the tunnel is assumed to be homogeneous and have permeability μ_0, permittivity ϵ, and conductivity σ, while the inside of the tunnel is air filled (μ_0, ϵ_0). The tunnel wall is denoted by the contour C, the circumference of the wire by the contour C_w, and the cross section of the tunnel (the region between C_w and C) by S. The interior cross section of the wire is denoted by S_w, and an outward normal unit vector to C or C_w will be denoted by $\bar{n}$ (or $\bar{n}'$, as appropriate).

By first assuming the wire to have a finite conductivity, we may express Katsenelenbaum's integral equation for this system as

$$\bar{E}(\bar{\rho}) = \frac{1}{2\pi}\left[\text{grad div} - \gamma_e^2\right] \cdot \int\int_{-\infty}^{\infty}\left(\frac{n^2(\bar{\rho}')}{N^2} - 1\right)\bar{E}(\bar{\rho}')K_0\left[u|\bar{\rho}-\bar{\rho}'|\right]dS' \quad (1)$$

where $\bar{\rho}=(x,y)$ and $\bar{\rho}'=(x',y')$ are points in the cross section, $\bar{E}$ is the vector electric field of the mode,

$$\gamma_e^2 = i\omega\mu_0(\sigma + i\omega\epsilon)$$

$$N^2 = (\epsilon - i\sigma/\omega)/\epsilon_0$$

$$n^2(\bar{\rho}) = \left[\epsilon(\bar{\rho}) - i\sigma(\bar{\rho})/\omega\right]/\epsilon_0$$

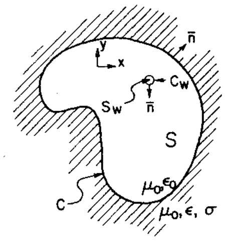

Fig. 1. Mine tunnel of arbitrary cross section.

$$u=(\gamma_e^2-\Gamma^2)^{1/2}, \qquad \mathrm{Re}(u)\geqslant 0$$

and Γ is the yet unknown propagation constant, appropriate to an assumed propagation factor of exp $(i\omega t-\Gamma z)$ along the tunnel. The quantity $n^2(\bar{\rho})$ is the (complex) index of refraction at any point in the wire, in the tunnel or in the tunnel wall, so that the factor $(n^2(\bar{\rho})/N^2-1)$ vanishes outside of $S+S_w$, and the integral in (1) is only over a finite cross section. K_0 is the usual modified Bessel function of the second kind, and the operators "grad" and "div" are understood to have $\partial/\partial z$ replaced by $-\Gamma$ according to the implied propagation factor referred to above. A brief derivation of (1) is given in Appendix B.

A variational expression can be formed from (1) in the usual way [6] by taking the scalar product of both sides with $(n^2(\bar{\rho})/N^2-1)\bar{E}^T(\bar{\rho})$ and integrating over x and y (once again the integral is actually only over $S+S_w$). The superscript "T" denotes a transpose field which can be obtained from $\bar{E}(\bar{\rho})$ by changing the sign of $E_z(\bar{\rho})$. In what follows, we will assume for simplicity that the wire is perfectly conducting, though this is not essential to the technique. To handle this case, we let $\sigma(\bar{\rho})\rightarrow\infty$ and $\bar{E}(\bar{\rho})\rightarrow 0$ in S_w in such a way that $\sigma(\bar{\rho})\bar{E}(\bar{\rho})$ passes over into a surface current distribution $\bar{J}_s(\bar{\rho})$ concentrated on C_w. The resulting variational expression is

$$\begin{aligned}\Delta\int_S \bar{E}^T\cdot\bar{E}\,dS=&\frac{\Delta^2}{2\pi}\int_S \bar{E}^T(\bar{\rho})\\ &\cdot\left[\text{grad div }-\gamma_e^2\right]\int_S \bar{E}(\bar{\rho}')K_0 dS'dS\\ &-\frac{i\Delta}{2\pi\omega\epsilon_0 N^2}\int_S \bar{E}^T(\bar{\rho})\cdot\left[\text{grad div}-\gamma_e^2\right]\oint_{C_w}\bar{J}_s(\bar{\rho}')K_0 dl'dS\\ &-\frac{i\Delta}{2\pi\omega\epsilon_0 N^2}\oint_{C_w}\bar{J}_s^T(\bar{\rho})\cdot\left[\text{grad div}-\gamma_e^2\right]\int_S \bar{E}(\bar{\rho}')K_0 dS'dl\\ &-\frac{1}{2\pi}\frac{1}{(\omega\epsilon_0 N^2)^2}\oint_{C_w}\bar{J}_s^T(\bar{\rho})\cdot\left[\text{grad div}-\gamma_e^2\right]\oint_{C_w}\bar{J}_s(\bar{\rho}')K_0 dl'dl\end{aligned}\qquad (2)$$

where $\Delta=(1/N^2-1)$ and the argument of the Bessel function has been omitted for brevity.

Since the wire is assumed to be thin, proximity effects are neglected and $\bar{J}_s$ consists only of a longitudinal component J_{sz}. But conservation of charge and boundary conditions on $\bar{E}$ imply that

$$J_{sz}|_{C_w}=\frac{i\omega\epsilon_0}{\Gamma}n\cdot E\Big|_{C_w}=\frac{i\omega\epsilon_0}{\Gamma}\bar{n}\cdot\bar{E}^T\Big|_{C_w}=-J_{sz}^T|_{C_w} \qquad (3)$$

and thus (2) reduces to

$$\begin{aligned}\Delta\int_S \bar{E}^T\cdot E\,dS=&\frac{\Delta^2}{2\pi}\int_S \bar{E}^T(\bar{\rho})\\ &\cdot\left[\text{grad div}-\gamma_e^2\right]\int_S \bar{E}(\bar{\rho}')K_0 dS'dS\\ &-\frac{\Delta}{2\pi N^2}\int_S \bar{E}^T(\bar{\rho})\cdot\text{ grad}\oint_{C_w}\left[\bar{n}'\cdot\bar{E}(\bar{\rho}')\right]K_0 dl'dS\\ &+\frac{\Delta}{2\pi N^2}\oint_{C_w}\left[\bar{n}\cdot E(\bar{\rho})\right]\text{ div}\int_S \bar{E}(\bar{\rho}')K_0 dS'dl\\ &+\frac{1}{2\pi}\frac{u^2}{N^4\Gamma^2}\oint_{C_w}\left[\bar{n}\cdot\bar{E}(\bar{\rho})\right]\oint_{C_w}\left[\bar{n}'\cdot\bar{E}(\bar{\rho}')\right]K_0 dl'dl.\end{aligned}\qquad (4)$$

As a trial function suitable for use when the frequency is sufficiently low, it seems appropriate to use the fields which would be present if the walls of the tunnel were perfectly conducting, i.e., the TEM fields $\bar{E}_0$, where

$$\bar{E}_0=-\nabla_t\Phi, \qquad \Phi|_{C_w}=V$$
$$\Phi|_C=0 \qquad (5)$$

and "t" denotes the transverse part of an operator, obtained by setting $\partial/\partial z=0$. If $\bar{E}_0$ is inserted into (4) as the trial field, some simplification occurs because $\bar{E}_0$ has no z-component:

$$\begin{aligned}\Delta\int_S \bar{E}_0^2 dS=&\frac{\Delta^2}{2\pi}\int_S \bar{E}_0(\bar{\rho})\cdot\left[\text{grad}_t\text{ div}_t-\gamma_e^2\right]\int_S \bar{E}_0(\bar{\rho})K_0 dS'dS\\ &-\frac{\Delta}{2\pi N^2}\int_S \bar{E}_0(\bar{\rho})\cdot\text{grad}_t\oint_{C_w}\left[\bar{n}'\cdot\bar{E}_0(\bar{\rho}')\right]K_0 dl'dS\\ &+\frac{\Delta}{2\pi N^2}\oint_{C_w}\left[\bar{n}\cdot\bar{E}_0(\bar{\rho})\right]\text{ div}_t\int_S \bar{E}_0(\bar{\rho}')K_0 dS'dl\\ &+\frac{1}{2\pi}\frac{u^2}{N^4\Gamma^2}\oint_{C_w}\left[\bar{n}\cdot\bar{E}_0(\bar{\rho})\right]\oint_{C_w}\left[\bar{n}'\cdot E_0(\bar{\rho}')\right]K_0 dl'dl.\end{aligned}\qquad (6)$$

Moreover, by virtue of the special form (5) of $\bar{E}_0$, the individual terms of (5) can be reworked into more familiar quantities. These manipulations are carried out in Appendix A.

From (6), (A1), (A8)–(A11), we obtain an equation for determining Γ in the thin-wire approximation:

$$-\Delta=\frac{-1}{2\pi}\left(\frac{C}{\epsilon_0}\right)\frac{u^2}{N^2\Gamma^2}\left[\ln\frac{ua}{2}+\gamma\right]-\Delta^2\Gamma^2N^2P$$
$$+\frac{\Delta^2}{2\pi}u^2\Gamma^2N^2Q-\frac{\Delta}{\pi}u^2R \qquad (7)$$

where $\gamma=0.577\cdots$ is Euler's constant, C is the capacitance per unit length the wire would exhibit if the tunnel walls were perfectly conducting, and

$$P=\frac{\epsilon_0}{CV^2}\int_S \Phi^2 dS \tag{8}$$

$$Q=\frac{\epsilon_0}{CV^2}\int_S \Phi(\bar{\rho})\int_S \Phi(\bar{\rho}')K_0 dS' dS \tag{9}$$

$$R=\frac{\epsilon_0}{CV^2}\oint_{C_w}\left[\bar{n}\cdot\bar{E}_0(\bar{\rho})\right]\int_S \Phi(\bar{\rho}')K_0 dS' dl. \tag{10}$$

Here V is the (arbitrary) potential difference associated with the trial field (5), and clearly drops out of (8)–(10).

As $\omega\to 0$, it can be seen that $N^2=0(1/\omega)$, $\gamma_e^2=0(\omega)$, $\Delta\to -1$, and with a certain amount of hindsight, $\Gamma=0(\omega)$[1], so that $u^2=0(\omega)$. Since (8)–(10) are $0(\omega^{\circ})$ as $\omega\to 0$, it can be seen that, for sufficiently low frequencies, (7) will reduce to

$$\Delta=\frac{1}{2\pi}\left(\frac{C}{\epsilon_0}\right)\frac{u^2}{N^2\Gamma^2}\left[\ln\frac{ua}{2}+\gamma\right] \tag{11}$$

a result which involves the tunnel geometry only through the capacitance C, which is tabulated for many different cross-sectional shapes (see e.g., [7]). Since C is generally only weakly dependent on the dimensions of the tunnel, it can be seen that the low-frequency behavior of Γ will only weakly depend on these dimensions as well.

As a check on all of this, we can compare with a case for which a closed-form eigenvalue equation for Γ is available, namely a wire centered in a circular tunnel of radius b. Now, the exact equation is mathematically the same as that for the Goubau line [8], and the low-frequency limit is found by taking small argument forms for the Bessel functions involved (see [8]) and results in

$$\Delta \ln\left(\frac{b}{a}\right)=\frac{u^2}{N^2\Gamma^2}\left[\ln\frac{ua}{2}+\gamma\right] \tag{12}$$

which agrees with (11) since for this case, $C/\epsilon_0=2\pi/\ln(b/a)$.

A further comparison can be made with the quasi-static limit for the offset wire in a circular tunnel obtained by Wait [3]. Calling the radius of the tunnel b, once again, and denoting $v=(-\omega^2\mu_0\epsilon_0-\Gamma^2)^{1/2}$, Wait's approximations up to and including eq. (13) of [3] can be summarized in our notation as

$$|vb|\ll 1 \qquad |N^2|\gg 1$$

and hence, $u\simeq\gamma_e$. Since our assumptions correspond to $|\gamma_e b|\ll 1$, we must take the small argument forms of the modified Bessel functions in eq. (13) of [3], and insert this value into eq. (7) of the same reference. The result is

$$\Gamma^2=ZY \tag{13}$$

where

$$Z=-\frac{i\omega\mu_0}{2\pi}\left(\ln\frac{\gamma_e a}{2}+\gamma\right)+Z_s \tag{14}$$

$$Y=2\pi i\omega\epsilon_0\left[\ln\frac{ab}{b^2-\rho_0^2}\right]^{-1} \tag{15}$$

where ρ_0 is the radial distance of the wire from the center of the tunnel, and Z_s is the axial impedance of the wire. Putting $Z_s=0$ in (14), making the previously indicated approximations in (11), and using the capacitance

$$\frac{C}{\epsilon_0}=2\pi\left[\ln\frac{b^2-\rho_0^2}{ab}\right]^{-1}$$

for the offset wire, we find that (13) and (11) give the same value for Γ.

Equation (11) can actually be cast in the suggestive form of (13) if we put

$$Z=\frac{i\omega\mu_0}{2\pi}\frac{u^2}{\Delta\gamma_e^2}\left(\ln\frac{ua}{2}+\gamma\right) \tag{16}$$

$$Y=i\omega C \tag{17}$$

in which case the real and imaginary parts of (16) can be related to the series resistance and inductance of the line, once the value of Γ is known. Equations (16) and (17) have the interesting interpretation that the low-frequency behavior of a wire in the tunnel is that of a transmission line whose shunt admittance is that which the wire would have if the tunnel walls were perfectly conducting (and thus is independent of the electric parameters of the rock) while the series impedance is that of the wire embedded in the rock with no tunnel surrounding it (and thus is independent of the shape of the tunnel)—see, e.g., [10].

III. Numerical Results

In order to compare results from this approximate analysis to exact results (which are only available for circular and elliptical cross sections), we need an expression for the distributed capacitance for these geometries. For the elliptical geometry shown in Fig. 2 (which includes the circular geometry as a special case) whose major and minor semi-axes are given by $A\equiv(d/2)\cosh\mu_1$ and $B\equiv(d/2)\sinh\mu_1$, respectively, and in which the wire is located at (μ_0,ϕ_0) in elliptic coordinates $(x+iy=(d/2)\cosh(\mu+i\phi))$, this capacitance is given by Morse and Feshbach [9] as

$$\frac{2\pi\epsilon_0}{C}=\ln\frac{d}{4a}+\mu_1-2\sum_{n=1}^{\infty}\frac{e^{-n\mu_1}}{n}\cdot\left[\frac{\cosh^2 n\mu_0\cos^2 n\phi_0}{\cosh n\mu_1}+\frac{\sinh^2 n\mu_0\sin^2 n\phi_0}{\sinh n\mu_1}\right]. \tag{18}$$

It is easily verified that (18) reduces to the expressions given in the previous section for the circular limit ($\mu_1\to\infty$, $d\to 0$, $d\exp(\mu_1)\to 4b$).

Results computed from (11) are compared with exact computations based upon the analysis of Wait and Hill [1] for a circular tunnel in Fig. 3. Here, the tunnel has radius $b=2$ m and contains a perfectly conducting thin wire with radius $a=1$ cm. The tunnel walls are characterized by

[1]For present purposes, $0(\omega)$ will be understood to include behavior such as $\omega\ln\omega$, $\omega(\ln\omega)^2$, etc.

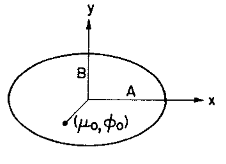

Fig. 2. Elliptical tunnel geometry.

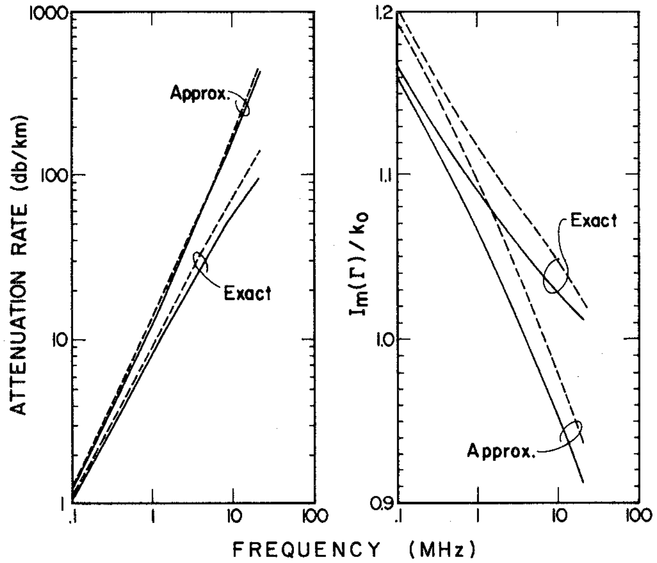

Fig. 3. Comparison of exact (Wait and Hill) and approximate (from (11)) solutions for attenuation rate and phase versus frequency, for a circular tunnel of radius $b=2$ m, with wire of radius $a=1$ cm, located at a radius ρ_0 from center of tunnel; $\epsilon=10\,\epsilon_0$, $\mu=\mu_0$, $\sigma=0.01$ mhos/m. $\rho_0=0$: ———. $\rho_0=1$ m: ----.

$\epsilon=10\epsilon_0$, $\mu=\mu_0$, and $\sigma=0.01$ mhos/m. The wire is positioned at a distance ρ_0 from the center of the tunnel. The solid lines correspond to a concentrically located wire, while the dashed lines refer to a wire offset $\rho_0=1$ m from the center of the tunnel. It can be seen that good agreement is obtained for typical tunnel parameters at frequencies of 100 kHz or below, and that results are adequate (within a factor of 2 for the attenuation) up to about 1 MHz. The phase constant is predicted accurately up to much higher frequencies. Fig. 4 compares results for the elliptic cross section using the present variational approach ((11)—solid lines), and those obtained using the two-dimensional quasi-static approximation in [4] (broken lines). Here, $(AB)^{1/2}=2$ m, $B/A=0.5$, and the wire is located on the major axis 0.4 m from the tunnel wall. Other parameters are the same as in Fig. 3. The two give indistinguishable values for attenuation below about 1 MHz, and thus we find similar agreement with exact results as in the circular case, because the quasi-static limit was found to give good results in this low-frequency range.

Fig. 5 demonstrates the improvement in these computations for a concentric circular tunnel obtained by keeping one higher degree of approximation in (7); that is, still neglecting Q, but retaining P and evaluating R approximately using the small argument form of K_0. Relevant parameters are the same as those for Fig. 3 with $\rho_0=0$.

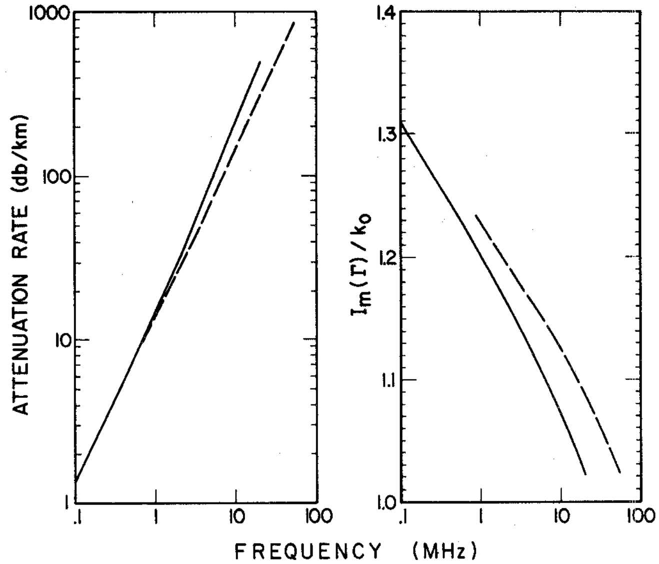

Fig. 4. Attenuation rate and phase versus frequency for an elliptical tunnel with $(AB)^{1/2}=2$ m, $B/A=0.5$, with wire of radius $a=1$ cm located on major axis 0.4 m from tunnel wall; $\epsilon=10\,\epsilon_0$, $\mu=\mu_0$, $\sigma=0.01$ mhos/m. Variational solution (from (11)): ———. Two-dimensional quasi-static solution (Seidel and Wait): ----.

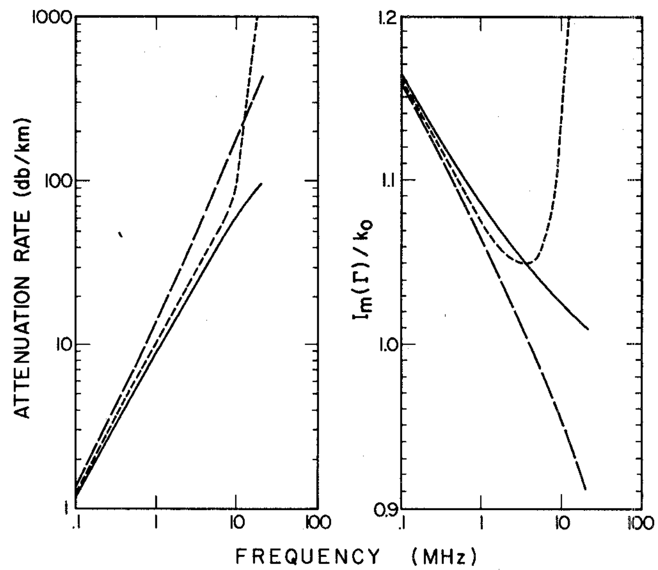

Fig. 5. Attenuation rate and phase versus frequency for a circular tunnel of radius $b=2$ m, with wire of radius $a=1$ cm located at center of tunnel ($\rho_0=0$); $\epsilon=10\,\epsilon_0$, $\mu=\mu_0$, $\sigma=0.01$ mhos/m. Zeroth-order variational (from (11)): ———. First-order variational (from (7), including P and R): --------. Exact (Wait and Hill): ———.

These calculations are elementary for the concentric circular case ($\rho_0=0$) and are omitted here. Here the exact solution is given by the solid lines, the zeroth-order variational solution from (11) by the broken lines, and the higher order solution by the dashed lines. The improvement is quite dramatic, especially in the attenuation, up to a few megahertz, but deteriorates rapidly thereafter. In view of the other idealizations involved in this problem (assumptions of an infinite homogeneous rock wall, no longitudinal irregularities, etc.) the effort necessary to compute P, Q, and R in more general situations does not seem justified in terms of the increased accuracy they provide.

Appendix A

In this Appendix, the terms of (6) are manipulated into convenient forms for use in a low-frequency approximation.

First, consider the left-hand side:

$$\begin{aligned}\int_S \bar{E}_0^2 dS &= \int_S \nabla_t\Phi\cdot\nabla_t\Phi dS \\ &= \int_S \nabla_t\cdot[\Phi\nabla_t\Phi]dS - \int_S \Phi\nabla_t^2\Phi dS \\ &= -V\oint_{C_w} \bar{n}\cdot\nabla_t\Phi dl = \frac{C}{\epsilon_0}V^2 \end{aligned} \tag{A1}$$

where C is the capacitance per unit length of the wire within a tunnel of identical cross section but with perfectly conducting walls. In deriving (A1), use is made of the divergence theorem and the fact that $\nabla_t^2\Phi=0$ in S.

Next, we have

$$\begin{aligned}&\int_S \bar{E}_0(\bar{\rho})\cdot[\text{grad}_t\,\text{div}_t - \gamma_e^2]\int_S \bar{E}_0(\bar{\rho}')K_0 dS' dS \\ &= \int_S \bar{E}_0(\bar{\rho})\cdot[\nabla_t^2 - u^2 - \Gamma^2 + \text{curl}_t\,\text{curl}_t]\int_S \bar{E}_0(\bar{\rho}')K_0 dS' dS \\ &= -2\pi\int_S \bar{E}_0^2 dS - \Gamma^2\int_S \bar{E}_0(\bar{\rho})\cdot\int_S \bar{E}_0(\bar{\rho}')K_0 dS' dS \\ &\quad + \int_S \bar{E}_0(\bar{\rho})\cdot\text{curl}_t\,\text{curl}_t\int_S \bar{E}_0(\bar{\rho}')K_0 dS' dS \end{aligned} \tag{A2}$$

since $(\nabla_t^2 - u^2)K_0 = -2\pi\delta(\bar{\rho}-\bar{\rho}')$. The first term is (A1); for the third term we have

$$\begin{aligned}&\int_S \bar{E}_0(\bar{\rho})\cdot\text{curl}_t\,\text{curl}_t\int_S \bar{E}_0(\bar{\rho}')K_0 dS' dS \\ &= -\int_S \nabla_t\Phi(\bar{\rho})\cdot\text{curl}_t\,\text{curl}_t\int_S \bar{E}_0(\bar{\rho}')K_0 dS' dS \\ &= -\int_S \nabla_t\cdot\left\{\Phi(\bar{\rho})\text{curl}_t\,\text{curl}_t\int_S \bar{E}_0(\bar{\rho}')K_0 dS'\right\}dS \\ &= V\oint_{C_w} \bar{n}\cdot\text{curl}_t\,\text{curl}_t\int_S \bar{E}_0(\bar{\rho}')K_0 dS' dl \\ &= V\int_{S_w} \text{div}_t\,\text{curl}_t\,\text{curl}_t\int_S \bar{E}_0(\bar{\rho}')K_0 dS' dl = 0. \end{aligned} \tag{A3}$$

The remaining integral in (A2) can be transformed by similar manipulations:

$$\begin{aligned}&\int_S \bar{E}_0(\bar{\rho})\cdot\int_S \bar{E}_0(\bar{\rho}')K_0 dS' dS \\ &= -\int_S \nabla_t\Phi(\bar{\rho})\cdot\int_S \bar{E}_0(\bar{\rho}')K_0 dS' dS \\ &= -\int_S \nabla_t\cdot\left[\Phi(\bar{\rho})\int_S \bar{E}_0(\bar{\rho}')K_0 dS'\right]dS \\ &\quad + \int_S \Phi(\bar{\rho})\nabla_t\cdot\int_S \bar{E}_0(\bar{\rho}')K_0 dS' dS. \end{aligned} \tag{A4}$$

Consider the first integral:

$$\begin{aligned}-\int_S \nabla_t\cdot&\left[\Phi(\bar{\rho})\int_S \bar{E}_0(\bar{\rho}')K_0 dS'\right]dS \\ &= V\oint_{C_w} \bar{n}\cdot\int_S \bar{E}_0(\bar{\rho}')K_0 dS' dl \\ &= V\int_{S_w} \nabla_t\cdot\int_S \bar{E}_0(\bar{\rho}')K_0 dS' dS \\ &= \int_{S_w} \Phi(\bar{\rho})\nabla_t\cdot\int_S \bar{E}_0(\bar{\rho}')K_0 dS' dS \end{aligned} \tag{A5}$$

defining $\Phi(\bar{\rho}) = V$ in S_w, since $\bar{E}_0 = 0$ there. Thus (A4) becomes

$$\int_S \bar{E}_0(\bar{\rho})\cdot\int_S \bar{E}_0(\bar{\rho}')K_0 dS' dS = \int_{S+S_w} \Phi(\bar{\rho})\nabla_t\cdot\int_S \bar{E}_0(\bar{\rho}')\cdot K_0 dS' dS$$

and since

$$\begin{aligned}\nabla_t\cdot\int_S \bar{E}_0(\bar{\rho}')K_0 dS' &= \int_S \nabla_t'\Phi(\bar{\rho}')\cdot\nabla_t' K_0 dS' \\ &= \int_S \nabla_t'\cdot[\Phi(\bar{\rho}')\nabla_t' K_0]dS' - \int_S \Phi(\bar{\rho}')\nabla_t'^2 K_0 dS' \\ &= -\int_{S+S_w} \Phi(\bar{\rho}')\nabla_t'^2 K_0 dS' \\ &= 2\pi\Phi(\bar{\rho}) - u^2\int_{S+S_w} \Phi(\bar{\rho}')K_0 dS'. \end{aligned} \tag{A6}$$

We obtain finally

$$\begin{aligned}&\int_S \bar{E}_0(\bar{\rho})\cdot\int_S \bar{E}_0(\bar{\rho}')K_0 dS' dS \\ &= 2\pi\int_{S+S_w} \Phi^2 dS - u^2\int_{S+S_w} \Phi(\bar{\rho})\int_{S+S_w} \Phi(\bar{\rho}')K_0 dS' dS \\ &= 2\pi\int_S \Phi^2 dS - u^2\int_S \Phi(\bar{\rho})\int_S \Phi(\bar{\rho}')K_0 dS' dS \end{aligned} \tag{A7}$$

upon neglecting the integrals over the wire cross section owing to our thin-wire assumption. Thus (A2) becomes

$$\begin{aligned}&\int_S \bar{E}_0(\bar{\rho})\cdot[\text{grad}_t\,\text{div}_t - \gamma_e^2]\int_S \bar{E}_0(\bar{\rho}')K_0 dS' dS \\ &\quad = -2\pi\frac{C}{\epsilon_0}V^2 - 2\pi\Gamma^2\int_S \Phi^2 dS + u^2\Gamma^2 \\ &\quad\quad \cdot\int_S \Phi(\bar{\rho})\int_S \Phi(\bar{\rho}')K_0 dS' dS. \end{aligned} \tag{A8}$$

Now, let us consider

$$\begin{aligned}&\int_S \bar{E}_0(\bar{\rho})\cdot\text{grad}_t\oint_{C_w}[\bar{n}'\cdot\bar{E}_0(\bar{\rho}')]K_0 dl' dS \\ &= \int_S \bar{E}_0(\bar{\rho})\cdot\oint_{C_w}[\bar{n}'\cdot\bar{E}_0(\bar{\rho}')]\nabla_t K_0 dl' dS \\ &= \oint_{C_w}[\bar{n}'\cdot\bar{E}_0(\bar{\rho}')]\int_S \bar{E}_0(\bar{\rho})\cdot\nabla_t K_0 dS dl' \\ &= -\oint_{C_w}[\bar{n}'\cdot\bar{E}_0(\bar{\rho}')]\,\text{div}_t'\int_S \bar{E}_0(\bar{\rho})K_0 dS dl' \end{aligned} \tag{A9}$$

which is precisely in the form of the third term on the right side of (6). By virtue of (A6), we may evaluate this term as

$$\oint_{C_w}\left[\bar{n}\cdot\bar{E}_0(\bar{\rho})\right]\nabla_t\cdot\int_S \bar{E}_0(\bar{\rho}')K_0 dS'\,dl$$
$$=2\pi V\oint_{C_w}\left[\bar{n}\cdot\bar{E}_0(\bar{\rho})\right]dl - u^2\oint_{C_w}\left[\bar{n}\cdot\bar{E}_0(\bar{\rho}')\right]$$
$$\cdot\int_S \Phi(\bar{\rho}')K_0 dS'\,dl - u^2 V\oint_{C_w}\left[\bar{n}\cdot\bar{E}_0(\bar{\rho})\right]\int_{S_w} K_0 dS'$$
$$=2\pi\frac{C}{\epsilon_0}V^2 - u^2\oint_{C_w}\left[\bar{n}\cdot\bar{E}_0(\bar{\rho})\right]\int_S \Phi(\bar{\rho}')K_0 dS'\,dl \quad \text{(A10)}$$

the thin-wire assumption having been invoked once more.

The final term of (6) can be evaluated within the thin-wire constraint by using the fact that the surface charge (i.e., $\bar{n}\cdot\bar{E}_0$) is nearly uniform over the boundary C_w of the wire, and can be reckoned constant. We then find, for a wire of radius a, that

$$\oint_{C_w}\left[\bar{n}\cdot\bar{E}_0(\bar{\rho})\right]\oint_{C_w}\left[\bar{n}'\cdot\bar{E}_0(\bar{\rho}')\right]K_0\,dl'\,dl$$
$$\cong -\left[\frac{CV}{\epsilon_0}\right]^2\left\{\ln\frac{ua}{2}+\gamma\right\} \quad \text{(A11)}$$

where $\gamma = 0.577\cdots$ is Euler's constant.

Appendix B

In this Appendix, a derivation of (1) is presented essentially as given in [5]. Consider a volume V of electrical parameters $\mu(\bar{r})$, $\epsilon(\bar{r})$, and $\sigma(\bar{r})$ embedded in an infinite, homogeneous region with constant parameters μ_1, ϵ_1, and σ_1. Source-free field solutions (e.g., fields of guided modes) can be considered to arise from the polarization currents $\bar{J}_e$ and $\bar{J}_m$ radiating in the absence of the body V:

$$\bar{J}_e = i\omega(\hat{\epsilon}-\hat{\epsilon}_1)\bar{E} \qquad \bar{J}_m = i\omega(\mu-\mu_1)\bar{H} \quad \text{(B1)}$$

where $\hat{\epsilon}\equiv\epsilon - i\sigma/\omega$ and $\hat{\epsilon}_1 = \epsilon_1 - i\sigma_1/\omega$. Even though $\bar{E}$ and $\bar{H}$ are as yet unknown, the fields can be derived from Hertz vectors $\bar{\Pi}_e$ and $\bar{\Pi}_m$ according to

$$\bar{E} = (\text{grad div} + k_1^2)\bar{\Pi}_e - i\omega\mu_1\,\text{curl}\,\bar{\Pi}_m \quad \text{(B2)}$$

$$\bar{H} = (\text{grad div} + k_1^2)\bar{\Pi}_m + i\omega\hat{\epsilon}_1\,\text{curl}\,\bar{\Pi}_e \quad \text{(B3)}$$

where

$$\bar{\Pi}_e = \frac{1}{i\omega\hat{\epsilon}_1}\int_V \bar{J}_e(\bar{r}')\frac{e^{-ik_1|r-r'|}}{4\pi|r-r'|}d\bar{r}' \quad \text{(B4)}$$

$$\bar{\Pi}_m = \frac{1}{i\omega\mu_1}\int_V \bar{J}_m(\bar{r}')\frac{e^{-ik_1|r-r'|}}{4\pi|r-r'|}d\bar{r}'. \quad \text{(B5)}$$

We have defined $k_1^2 = \omega^2\mu_1\hat{\epsilon}_1$ and Im $(k_1)<0$ for $\sigma_1>0$. If V, μ, ϵ, and σ are translationally invariant in the z-direction (i.e., V is some cylinder with constant cross-section S) and mode fields of the form $\bar{E}=\bar{E}(\bar{\rho})e^{-\Gamma z}$ and $\bar{H}=\bar{H}(\bar{\rho})e^{-\Gamma z}$ are assumed, with $\bar{\rho}=(x,y)$, then the z' integrations in (B4) and (B5) can be done in closed form provided $|\text{Re}\,\Gamma|<|\text{Im}\,k_1|$:

$$\bar{\Pi}_e = \frac{e^{-\Gamma z}}{2\pi}\int\left(\frac{\hat{\epsilon}(\rho')}{\hat{\epsilon}_1\bar{\rho}'}-1\right)\bar{E}(\rho')K_0\left[u|\bar{\rho}-\bar{\rho}'|\right]d\bar{\rho}' \quad \text{(B6)}$$

$$\bar{\Pi}_m = \frac{e^{-\Gamma z}}{2\pi}\int\left(\frac{\mu(\bar{\rho}')}{\mu_1}-1\right)\bar{H}(\bar{\rho}')K_0\left[u|\bar{\rho}-\bar{\rho}'|\right]d\bar{\rho}'. \quad \text{(B7)}$$

We have put $u=(-k_1^2-\Gamma^2)^{1/2}$ and Re $(u)>0$. Here K_0 is the modified Bessel function of the second kind. Upon putting $\mu(\bar{\rho})\equiv\mu_1$, $\bar{\Pi}_m$ vanishes and the integral (1) follows at once from (B2) by taking the observation point $\bar{\rho}$ inside S.

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